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Relaxation of metastable states in finite mean-field kinetic Ising systems

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Abstract. We present simulational data and a theoretical analysis of the stochastic order parameter relaxation in finite mean-field Ising systems. Contrary to infinite mean-field systems the finite ones exhibit qualitatively the same kind of relaxation as systems with finite interaction ranges. We explore the equivalence of different versions of the kinetic Ising model in the mean-field case and study the lifetime of metastable states. We focus on the vicinity of the mean-field spinodal and present a phenomenological finite-size-scaling theory for the lifetime of the metastable state.

1. Introduction

The kinetics of first-order phase transitions as an example of the statistical mechanics of systems far from equilibrium still pose a lot of fundamental problems (Binder 1987, Gunton *et al* 1983, Klein and Unger 1983, Metiu *et al* 1979, Penrose and Lebowitz 1979). There are some phenomenological theories that can be dealt with in approximations. In the unstable region, time-dependent Ginzburg-Landau theory for non-conserved order parameter or Cahn-Hilliard-Cook theory for conserved one, for example, or close to the coexistence curve for instance, the Becker-Döring theory of nucleation. On the other hand, there are simulational studies of model systems where the simulated evolution equation is valid in the metastable and the unstable region, molecular dynamics studies of Lennard-Jones systems (Abraham *et al* 1982) or Monte Carlo studies of lattice spin systems (Binder 1979). Most prominent among the last are the kinetic Ising models first studied by Glauber (1963). The critical dynamics of these systems in the mean-field approximation was analysed by Suzuki (Suzuki and Kubo 1968). We want to look at the mean-field version of these models with a focus on the properties of the stochastic process these models generate on the global magnetisation per spin of the system, and to understand the relaxation of the magnetisation far from equilibrium. Specifically we consider processes where we prepare a system with all spins down (external field is minus infinity) and then reverse the external field to some positive value (see figure 1). We thereby force the system to relax into equilibrium by passing an intermediate metastable state. Figure 2 shows a qualitative sketch of the generalised free energy $f_{T,H}(m)$ along the broken path in figure 1. Section 2 gives the general theoretical framework for the analysis of this process. In § 3 we explore the question of equivalence between different versions of kinetic Ising models, § 4 presents numerical data on the system size dependence of the stochastic properties and in § 5 we give our conclusions.

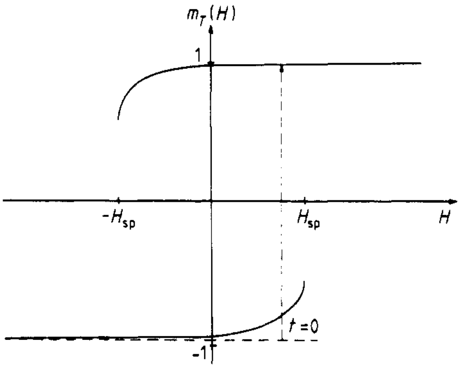

Figure 1. Mean-field equation of state $m_T(H)$ for a ferromagnet below the Curie temperature and the kind of relaxation paths we want to analyse.

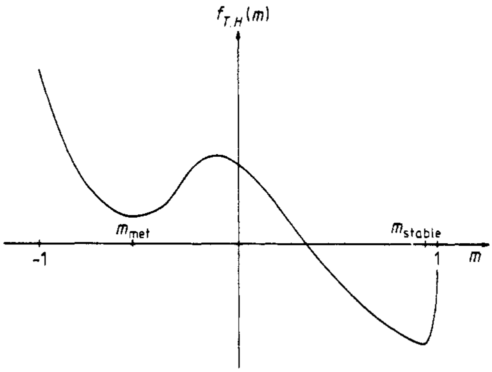

Figure 2. Qualitative sketch of the generalised free energy for $T < T_c$ and external field $0 < H < H_{sp}$.

2. The stochastic mean-field system

Let us consider spin systems described by the equivalent neighbour Ising Hamiltonian with ferromagnetic coupling (Domb and Dalton 1966, Griffiths *et al* 1966, Heermann *et al* 1982)

$$\mathscr{H}(s_1, \ldots, s_N) = -J_q \sum_{i=1}^{N} s_i \sum_{j \in n(i)} s_j - \mu\beta H \sum_{i=1}^{N} s_i =: \sum_{i=1}^{N} s_i E_i \qquad J_q := \frac{6}{q} J_{\text{Ising}} > 0 \tag{1}$$

where $n(i)$ is the set of spins interacting with spin i and q is the number of these spins. The corresponding kinetic Ising models for single spin flip are given by a microscopic master equation on the states $(s_1, \ldots, s_N)$ of the system:

$$\begin{aligned}\frac{\partial}{\partial t} P(s_1, \ldots, s_N, t) &\\ &= -\sum_{j=1}^{N} W(s_j \to -s_j) P(s_1, \ldots, s_N, t)\\ &\quad + \sum_{j=1}^{N} W(-s_j \to s_j) P(s_1, \ldots, -s_j, \ldots, s_N, t)\end{aligned} \tag{2}$$

with the condition of detailed balance imposed on the transition rates:

$$W(s_j \to -s_j) P_{\mathrm{eq}}(s_1, \ldots, s_N, t) = W(-s_j \to s_j) P_{\mathrm{eq}}(s_1, \ldots, -s_j, \ldots, s_N, t). \tag{3}$$

Two frequently used kinetic Ising models are given by the transition rates:

$$W_{\mathrm{G}}(s_j, E_j) = \frac{1}{2\tau}(1 - \tanh(\beta s_j E_j)) \qquad \text{Glauber} \tag{4a}$$

$$W_{\mathrm{M}}(s_j, E_j) = \frac{1}{\tau}\min(1, \exp\{-2\beta s_j E_j\}) \qquad \text{Metropolis} \tag{4b}$$

where $\beta = 1/k_{\mathrm{B}}T$ and τ is a constant setting the time scale; s_j and E_j denoting spin and local field before the spin flip. In the mean-field system the set $n(i)$ of spins interacting with spin i is given by all the other spins of the lattice so that

$$s_j E_j^{\mathrm{mf}} = (J_{N-1} N m + \mu\beta H) s_j - J_{N-1}. \tag{5}$$

Using $\delta m = -2s_j/N$ one can rewrite the transition probabilities as

$$\tilde{W}_{\mathrm{G}}(\delta m, m) = \frac{1}{2\tau}\left[1 + \tanh\left(\frac{T_{\mathrm{c}}}{2T}\frac{N^2}{N-1} m\delta m + \tfrac{1}{4}hN\delta m + \frac{T_{\mathrm{c}}}{T}\frac{1}{N-1}\right)\right] \tag{6a}$$

$$\tilde{W}_{\mathrm{M}}(\delta m, m) = \frac{1}{\tau}\min\left[1, \exp\left(\frac{T_{\mathrm{c}}}{T}\frac{N^2}{N-1} m\delta m + \tfrac{1}{2}hN\delta m + 2\frac{T_{\mathrm{c}}}{T}\frac{1}{N-1}\right)\right] \tag{6b}$$

with the abbreviations

$$k_{\mathrm{B}}T_{\mathrm{c}} = J_{N-1}(N-1) \tag{7a}$$

$$h = \frac{2\mu_{\mathrm{B}}H}{k_{\mathrm{B}}T}. \tag{7b}$$

Summing (2) over all states $\{s\}$ to fixed magnetisation per spin m and using (6), one now easily derives

$$\tau\frac{\partial}{\partial t}P(m,t) = -\frac{N}{2}[g(m) + r(m)]P(m,t) + \frac{N}{2}g\left(m - \frac{2}{N}\right)P\left(m - \frac{2}{N}, t\right) + \frac{N}{2}r\left(m + \frac{2}{N}\right)P\left(m + \frac{2}{N}, t\right) \tag{8}$$

with

$$g(m) := (1-m)\tilde{W}\left(\frac{2}{N}, m\right) \qquad g(1) = 0 \tag{9a}$$

$$r(m) := (1+m)\tilde{W}\left(-\frac{2}{N}, m\right) \qquad r(-1) = 0. \tag{9b}$$

Rewriting (8) as

$$\tau\frac{\partial}{\partial t}P(m,t) = (E-1)r(m)P(m,t) + (E^{-1}-1)g(m)P(m,t) \tag{8'}$$

where

$$E = \exp\left(\frac{2}{N}\frac{\partial}{\partial m}\right) \qquad \text{translation by } 2/N \tag{10a}$$

$$E^{-1} = \exp\left(-\frac{2}{N}\frac{\partial}{\partial m}\right) \qquad \text{translation by } -2/N \tag{10b}$$

and using the Kramers–Moyal expansion (the jump distance $2/N$ is the parameter), one arrives at the diffusion approximation to the one-step-process master equation (8):

$$\tau\frac{\partial}{\partial t}P(m,t)=-\frac{\partial}{\partial m}[D_1(m)P(m,t)]+\frac{1}{2}\frac{\partial^2}{\partial m^2}[D_2(m)P(m,t)]. \tag{11}$$

We will confine ourselves to the standard Kramers–Moyal expansion but for a quantitative comparison of the descriptions based on the master equation and Fokker–Planck equation respectively it would be worthwhile to exploit the consequences of a refined approximation to the master equation (Hänggi *et al* 1984).

The drift coefficient in (11) is given by

$$D_1(m)=g(m)-r(m)=:-U'(m) \tag{12a}$$

defining the drift potential $U(m)$. The diffusion coefficient is

$$D_2(m)=\frac{2}{N}[g(m)+r(m)]. \tag{12b}$$

In the thermodynamic limit the diffusion coefficient vanishes and the evolution of the order parameter (derivable from (8) or equivalently from (11))

$$\tau\frac{\partial}{\partial t}\langle m\rangle=-\langle U'(m)\rangle \tag{13}$$

becomes deterministic

$$\tau\frac{\partial}{\partial t}\langle m\rangle=-U'(\langle m\rangle). \tag{14}$$

The master equation (8) also offers a way to calculate the lifetime of metastable states. Imagine a system starting in or close to a metastable state. One would say that the system is still in this metastable state as long as it stays in its basin of attraction, and that the metastable state has decayed when the system has crossed the barrier to another metastable or to the stable state. Clearly this is a question of mean first passage times and for the given one-step process (8) these can be calculated as (Weiss 1967, Gillespie 1981, Seshadri *et al* 1980, van Kampen 1981)

$$T_1(m_0\to m)=\sum_{m'=m_0}^{m-2/N}\frac{2}{Ng(m')P_{\text{eq}}(m')}\sum_{x=-1}^{m'}P_{\text{eq}}(x). \tag{15}$$

Here $P_{\text{eq}}(m)$ denotes the equilibrium distribution for the grand canonical ensemble. In the following chapter we will explore the implications of equations (8)–(15).

3. Equivalence of kinetic Ising models

In spite of the fact that we can only treat the mean-field case analytically, we think that some of our conclusions will qualitatively also hold for medium range interaction models (Paul *et al* 1989).

3.1. Static properties

For the mean-field system the partition function

$$Z=\sum_{\{s\}}\exp\{-\beta\mathcal{H}(\{s\})\} \tag{17}$$

can be partially summed to yield:

$$Z = \sum_m \exp\{-\beta N f(m)\} \tag{17'}$$

with

$$f(m) = -J_{N-1} N m^2 - \mu_B H m + J_{N-1} \frac{1}{N\beta} \ln \#(m) \tag{18}$$

and

$$\#(m) = \frac{N!}{[(N/2)(1-m)]![(N/2)(1+m)]!}. \tag{19}$$

$\#(m)$ is the number of states for a given magnetisation m and $f(m)$ is the so-called generalised free energy. The function $f(m)$ for temperatures below the critical temperature has the well known double well shape with (for positive H) a metastable minimum at $m_{\text{met}} < 0$, an unstable maximum at $m_{\text{uns}} < 0$ and a stable equilibrium state at $m_{\text{eq}} \leqslant 1$. The thermodynamic properties of the system as well as the quasistatic 'thermodynamic' properties of the metastable state depend only on the location of the stable, respectively metastable, minimum as a function of T and H. Here we anticipate that this conclusion is corroborated from studying the kinetic properties as well. A treatment of the master equation yields these positions as the locations of the corresponding minima in the respective drift potentials. In the thermodynamic limit these are for the two choices (4*a*) and (4*b*)

$$U_G = \tfrac{1}{2}m^2 - \frac{T}{T_c} \ln \cosh\left(\frac{T_c}{T} m + \tfrac{1}{2}h\right) \tag{20a}$$

$$\begin{aligned} U_M &= m + \tfrac{1}{2}m^2 - \frac{T}{2T_c}\left(1 - m + \frac{T}{2T_c}\right) \exp\left\{\frac{2T_c}{T} m + h\right\} \qquad & m < -\frac{T}{2T_c} h \\ U_M &= -m + \tfrac{1}{2}m^2 - \frac{T}{2T_c}\left(1 + m + \frac{T}{2T_c}\right) \exp\left\{-\frac{2T_c}{T} m - h\right\} \qquad & m \geqslant \frac{-T}{2T_c} h \end{aligned} \tag{20b}$$

where $h = 2\beta\mu_B H$ will be used henceforth as a dimensionless parameter. Figures 3 and 4 show the drift potentials and the generalised free energy (also taken in the thermodynamic limit) for a temperature $T = \frac{4}{9}T_c$ and $h = \frac{1}{2}h_{\text{sp}}$ and h slightly exceeding

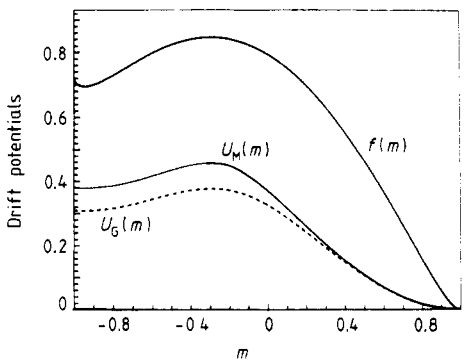

Figure 3. Kinetic potentials and generalised free energy for $T = \frac{4}{9}T_c$ and $h = \frac{1}{2}h_{\text{sp}}$.

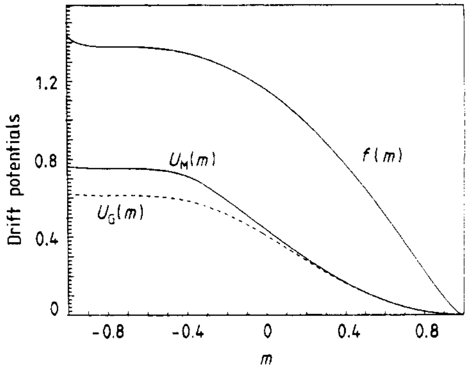

Figure 4. Kinetic potentials and generalised free energy for $T = \frac{4}{9}T_c$ and $h = 1.007\ h_{sp}$.

the spinodal field respectively. The spinodal value of the magnetic field is defined as the value where the potentials change from a double well to a single well structure, that is where the metastable minimum and the barrier combine to form a saddle point. In spite of the apparent differences between the three functions, all give the same condition for the location of their extrema and are thus equivalent regarding the thermodynamic properties. The behaviour of the three potentials around the stable equilibrium well that lies very close to $m = 1$ is shown in detail in figure 5. The h dependence of the location of the metastable minimum and the barrier are shown in figure 6. Both curves end at the spinodal field $h_{sp} \simeq 1.429\ 25$ separating the metastable from the unstable region.

3.2. Kinetic properties

The first thing to note is that according to equation (13) or (14) the evolution of the order parameter is not given by $\partial m/\partial t \sim \partial f/\partial m$ as usually assumed. This is a consequence of the magnetisation dependence of the diffusion which always leads to a difference between the potential governing the statics, $f(m)$, and the potential governing the kinetics, $U_M(m)$ or $U_G(m)$ (van Kampen 1981). This remains true for a field theoretic Ginsburg–Landau-like treatment of the inhomogeneous system (Paul 1988). A relation $\partial m/\partial t \sim \partial f/\partial m$ is only found to leading order for $T \to T_c$, as will be discussed in detail below.

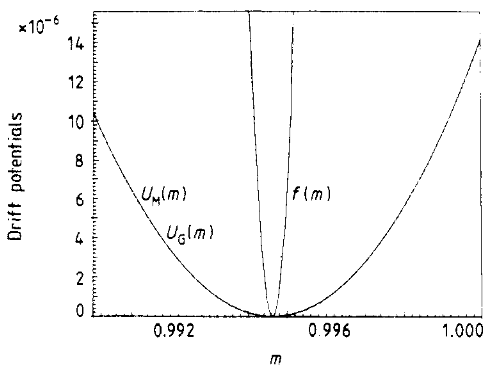

Figure 5. Behaviour of the different potentials near the stable minimum.

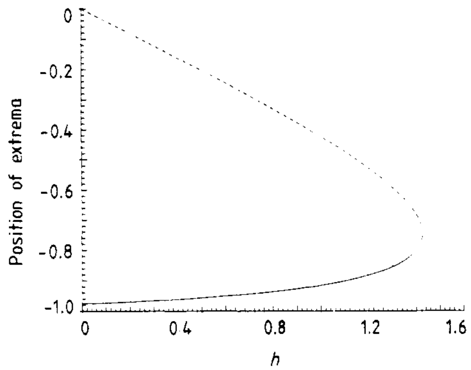

Figure 6. Location of the metastable minimum (full curve) and the barrier (broken curve) as a function of the applied field h for $T = \frac{4}{9}T_c$.

Furthermore the qualitative features of a diffusion process in a double well potential will be the same for both choices of transition probabilities, whereas quantities depending on the global form of the drift potential will differ. One such quantity is the mean first passage time from the metastable to the stable region, which is a measure for the lifetime of the metastable state. As long as the lifetime of the metastable state is large compared with the equilibrium relaxation times, the mean first passage times will not sensitively depend on where exactly in the metastable well the system starts. Figure 7 shows the mean first passage time to magnetisations $m \in [-1, 1]$ starting from $m_0 = -1$. The MC data were obtained by simulating equation (8) with Metropolis transition rates and the curves by evaluating (15) for the two choices (4*a*) and (4*b*). The MC data were obtained for a magnetic field slightly larger than the spinodal field in order to get the whole relaxation process into the time window of the first 100 Monte Carlo steps. But as the transition across the spinodal field is absolutely continuous for finite mean-field systems, the shape of the curve is the same as in the metastable region. Thus the lifetime of the metastable state would be definable as the height of the plateau in the mean first passage time curves and clearly depends on the version of the kinetic Ising model that one chooses. Quantities depending only on the local structure, however, are not sensitive to the chosen model. Such quantities are obtainable by an expansion of the drift potentials around a given value $\bar{m}$, for example $\bar{m} = m_{st}$ or

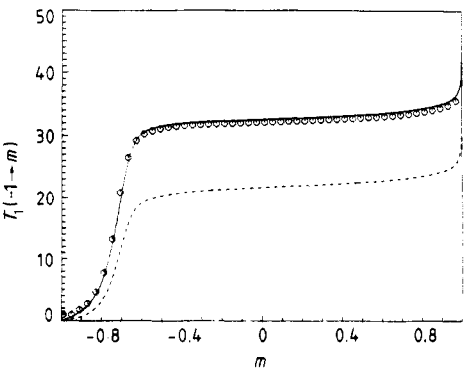

Figure 7. Model dependence of the mean first passage time (for explanations, see text): ○, MC data; ——, Metropolis; – – –, Glauber.

$\bar{m} = m_{\mathrm{met}}$ and linear response theory, or $\bar{m} = 0$ and critical dynamics. For linear response theory this is trivial because the extrema of the drift potentials agree and all differences can be absorbed in the time scale. (For an analysis of the Ising case see Müller-Krumbhaar and Binder (1973).) Near T_c one arrives at the Landau theory by expanding $f(m)$ up to order m^4. Since in Landau theory $m_{\mathrm{eq}} \sim (1-(T/T_c))^{1/2}$ and one is only interested in a region of the size of the double well structure, this amounts to an expansion up to order $(1-(T/T_c))^2$. In order to retain the double well shape the external magnetic field has to be smaller than the spinodal field given by

$$h_{\mathrm{sp}} = 2\frac{T_c}{T}\left(1-\frac{T}{T_c}\right)^{1/2} + \ln\frac{1-(1-(T/T_c))^{1/2}}{1+(1-(T/T_c))^{1/2}} \tag{21}$$

or for $T \simeq T_c$

$$h_{\mathrm{sp}} \simeq \frac{4}{3}\left(1-\frac{T}{T_c}\right)^{3/2}. \tag{21$'$}$$

Thus mh is of the order $(1-(T/T_c))^2$ and all higher terms have to be discarded in the expansions. For the three potentials one obtains

$$f(m) = \frac{1}{2}\left(1-\frac{T}{T_c}\right)m^2 - \tfrac{1}{2}hm + \tfrac{1}{12}m^4 = U_{\mathrm{G}}(m) = \tfrac{1}{2}U_{\mathrm{M}}(m). \tag{22}$$

So the free energy and the Glauber transition rate yield the same driving potential near T_c, whereas the Metropolis transition rate leads to a rescaling of the timescale by a factor of 2.

4. Relaxation in finite mean-field systems

If not stated explicitly otherwise, the following data were obtained by a Monte Carlo simulation of equation (8).

4.1. The non-linear relaxation function and recrossing events

The non-linear relaxation function is defined (Binder 1973) as

$$\phi(t) := \frac{m(t) - m(\infty)}{m(0) - m(\infty)}. \tag{23}$$

Figure 8 shows this function for a few selected linear system sizes $L = N^{1/3}$. The full curve is an integration of equation (14). The plateau that the relaxation function develops for linear system size $L \to \infty$ can be regarded as a precursor of metastability. In fact, the length of such a plateau is one of the possible ways to define the lifetime of a metastable state in finite systems (Binder 1973). As one approaches the thermodynamic limit the plateau gets longer and the relaxation down from the plateau gets steeper. Both effects arise from the same origin. With increasing system size the probability for fluctuations around the mean path decreases so that there are less large fluctuations driving the system quickly across the flat part in the drift potential (see figure 4) or holding it back on the plateau against the drift. A sensitive measure for this vanishing of stochasticity as well as for the occurrence and position of metastable states are the recrossing events (see Paul and Heermann (1988) for a discussion of the

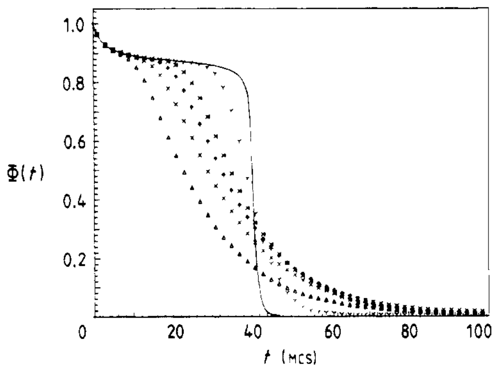

Figure 8. The non-linear relaxation function for a few selected system sizes. Values of L are: 40(Y), 21(⋊), 17(♆), 13(×), 9(△). Full curve, $L \to \infty$. Time is measured in Monte Carlo steps per site (MCS).

recrossing events in connection with systems with a finite interaction range). If one imagines some oscillatory relaxation path, the system crosses any fixed magnetisation for several times in the positive as well as the negative direction. The first passage in the positive direction defines a first passage time and each pair of crossing in the negative direction and return crossing in the positive direction defines a recrossing event. Figure 9 shows the distribution of the recrossing events as a function of the magnetisation. We consider the peak position as a good criterion to define where a metastable state occurs. The peak height decreases as the relaxation gets more and more deterministic as one approaches the thermodynamic limit. Furthermore, the shift in the peak position indicates a finite-size effect in the drift potential.

4.2. *The mean first passage time*

For fixed $h > h_{\text{sp}}$ the mean first passage time approaches a limiting curve for $L \to \infty$ (see figure 10). This curve is most conveniently found from the fact that the relaxation becomes deterministic. Equation (14) then directly yields

$$T_1(-1 \to m) \xrightarrow[L \to \infty]{} \int_{-1}^{m} \frac{\mathrm{d}x}{-U'(x)} \tag{24}$$

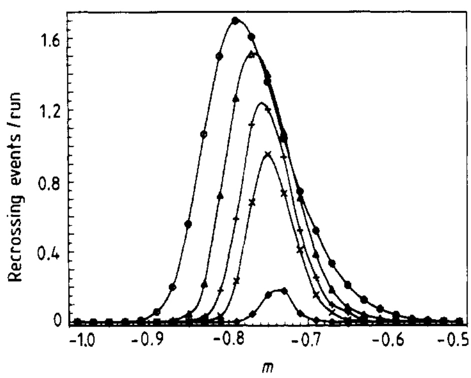

Figure 9. Distribution of the recrossing events for a few selected system sizes. Values of L are: 9(○), 13(△), 17(+), 21(×), 40(◇).

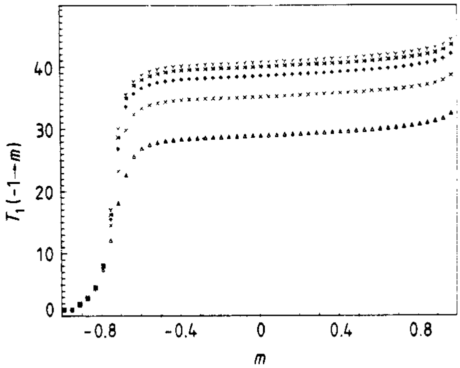

Figure 10. Monte Carlo results for the mean first passage time at $h = 1.007\ h_{sp}$ for a few selected system sizes. Symbols as in figure 8.

(simply look at the full curve in figure 8 from the left.)

The lifetime of the metastable state can be defined as the height of the plateau.

Figure 11 shows the mean first passage time for a system set up with $m_0 = -1$ to reach $m = 0$ as a function of the applied magnetic field for various system sizes. The data are the evaluation of equation (15) which for finite system size gives a continuous function across the spinodal field. In the thermodynamic limit the behaviour of the mean first passage time differs in the metastable and the unstable region. In the metastable region, $h < h_{sp}$, the generalised free energy has a double well shape, i.e. it is not a monotonically decreasing function of the magnetisation. If one writes out (15) more explicitly

$$T_1(-1 \to \infty) = \sum_{m'=-1}^{m-2/N} \frac{2}{Ng(m')} \sum_{x=-1}^{m'} \exp\{\beta N[f(m') - f(x)]\}$$

one sees that there appear terms proportional to $\exp\{\beta N[f(m') - f(x)]\}$.

As soon as m lies to the right of $m_{met}, f(m') - f(x)$ is positive for some m' and the mean first passage time diverges exponentially. If m lies in the stable well one sees that the lifetime of the metastable state will be asymptotically proportional to the dominant term, that is

$$\tau \propto \exp\{\beta N[f(m_{uns}) - (m_{met})]\}. \qquad (25)$$

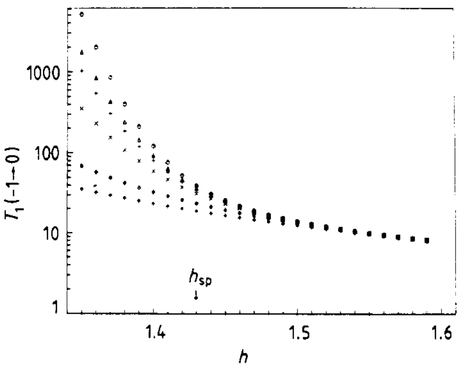

Figure 11. Finite-size dependence of the lifetime of the metastable state as a function of the applied field. Values of N are: 1000(○), 800(△), 700(+), 500(×), 200(◇), 100(†).

Here we recover a finding of Griffiths *et al* (1966) for the relaxation of metastable states in a mean-field kinetic Ising model. One would get the same expression out of thermodynamic fluctuation theory (Becker and Döring 1935) or transition state theory (see for instance Kramers 1940) if one assumed $f(m_{\text{uns}})$ and $f(m_{\text{met}})$ to be the free energies of thermodynamically stable states. On the unstable side one encounters a critical divergence of the relaxation time as $\tau \propto (h-h_{\text{sp}})^{-1/2}$ if one approaches the spinodal (Binder 1973).

4.3. *Finite-size scaling of the relaxation near mean-field spinodals*

We found that in the metastable region the relaxation time behaved asymptotically as $\tau \propto \exp\{N\Delta\tilde{f}\}$; $\Delta\tilde{f} = \beta[f(m_{\text{uns}}) - f(m_{\text{met}})] = \tilde{f}(m_{\text{uns}}) - \tilde{f}(m_{\text{met}})$. In order to understand this finite-size effect near the spinodal more precisely, we need to study how $\Delta\tilde{f}$ varies with $h_{\text{sp}} - h$. For this purpose we note that in the thermodynamic limit the generalised free energy can be written as follows:

$$\tilde{f}(m) = \frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right) - \frac{1}{2}\frac{T_c}{T}m^2 - mh. \tag{26}$$

Now the extrema in figure 4 are found from $f'(m) = \frac{1}{2}\ln(1+m) - \frac{1}{2}\ln(1-m) - m(T_c/T) - h = 0$. Solving this equation and inserting the two relevant solutions into the above free energy expression, $\Delta\tilde{f}$ follows. We want to solve this problem in the vicinity of the spinodal only. The spinodal is found from

$$f''(m) = \frac{1}{2}\frac{1}{1+m} + \frac{1}{2}\frac{1}{1-m} - \frac{T_c}{T} = 0$$

$$\Rightarrow m_{\text{sp}} = (\pm)\left(1 - \frac{T}{T_c}\right)^{1/2}$$

$$h_{\text{sp}} = \tfrac{1}{2}\ln(1+m_{\text{sp}}) - \tfrac{1}{2}\ln(1-m_{\text{sp}}) - m_{\text{sp}}\frac{T_c}{T}.$$

Writing then $m = m_{\text{sp}} + x$

$$\ln(1+m) \simeq \ln(1+m_{\text{sp}}) + \frac{x}{1+m_{\text{sp}}} - \frac{1}{2}\frac{x^2}{(1+m_{\text{sp}})^2}$$

$$\ln(1-m) \simeq \ln(1-m_{\text{sp}}) - \frac{x}{1-m_{\text{sp}}} - \frac{1}{2}\frac{x^2}{(1-m_{\text{sp}})^2}$$

we find

$$h - h_{\text{sp}} = x^2\left(\frac{T_c}{T}\right)^2 m_{\text{sp}} \qquad x = \pm\frac{T}{T_c}\sqrt{(h_{\text{sp}} - h)/|m_{\text{sp}}|}$$

and

$$\Delta\tilde{f} = 4\left(\frac{T}{T_c}\right)(h_{\text{sp}} - h)^{3/2}|m_{\text{sp}}|^{-1/2}. \tag{27}$$

Thus we conclude that the relaxation time behaves asymptotically as

$$\tau \propto \exp\{N\Delta\tilde{f}\} \simeq \exp\left\{N(h_{\text{sp}} - h)^{3/2}4\left(\frac{T}{T_c}\right)|m_{\text{sp}}|^{-1/2}\right\}. \tag{28}$$

The combination of N and $h_{sp}-h$ that describes the finite-size rounding of the relaxation time on the metastable side of the spinodal is hence given by $N(h_{sp}-h)^{3/2}$.

We now make the assumption that an analogous combination also enters on the unstable side of the transition, and thus write down the scaling ansatz

$$\tau(h, N) = (h - h_{sp})^{-1/2}\tilde{\tau}(N(h - h_{sp})^{3/2}). \tag{29}$$

The requirement that at $h = h_{sp}$ the singular factor $(h - h_{sp})^{-1/2}$ cancels out means that the scaling function $\tilde{\tau}(\mathfrak{z})$ for small arguments $\mathfrak{z}$ must behave as $\tilde{\tau}(\mathfrak{z}) \propto \mathfrak{z}^{1/3}$ and hence we predict

$$\tau(h_{sp}, N) \sim N^{1/3}. \tag{30}$$

One can test this scaling theory by analysing (15) near $h = h_{sp}$. In figure 12 we plot the scaling function $\tau(h, N)|h - h_{sp}|^{1/2}$ against $N\,|h - h_{sp}|^{3/2}$. The straight line indicates a slope of $\frac{1}{3}$. In the plotted range of magnetic field values $h = 1.429$–1.43 with $h_{sp} = 1.429\,25$ at $T/T_c = \frac{4}{9}$ we find perfect agreement with the predicted scaling.

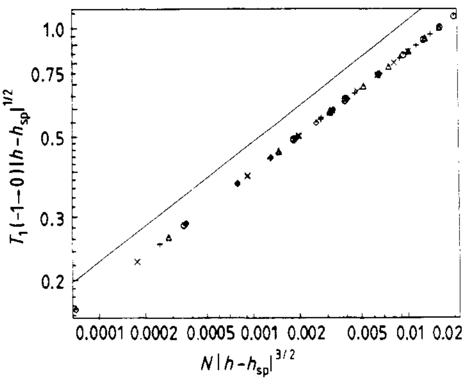

Figure 12. Finite-size-scaling plot of the lifetime of the metastable state near the spinodal. Symbols as in figure 11.

5. Conclusions

We have analysed the mean-field theory for the kinetics of first-order phase transitions in the kinetic Ising model. The version of the kinetic Ising model used, as specified by the transition rates chosen in the master equation, influences quantities depending on the global properties of the drift potentials that are derivable from the transition rates. One such quantity is the lifetime of the metastable state that can be defined as the mean first passage time from the metastable well of the drift potential to the stable one. The different versions of the kinetic Ising model are equivalent regarding localised relaxation processes as those encountered in linear response theory or critical dynamics. All differences can be absorbed in the timescale. Finally we presented a finite-size-scaling theory for the lifetime of the metastable state in the vicinity of the mean-field spinodal, where a critical divergence occurs. This phenomenological finite-size-scaling theory is confirmed by our numerical calculations.

We believe that the fact that quantities like the lifetime of the metastable state that depend on global properties of the diffusion process will be influenced by the choice of transition rates should carry over to the kinetic Ising model with short- or medium-range interactions. These systems will be analysed elsewhere (Paul *et al* 1989).

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